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Nonperturbative confinement in quantum chromodynamics. II. Mandelstam's gluon propagator

D. Atkinson, P. W. Johnson,[a)] and K. Stam
Institute for Theoretical Physics, University of Groningen, P. O. Box 800, 9700 A V Groningen, The Netherlands

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It is shown that Mandelstam's approximate equation for the gluon propagator has a solution with very singular infrared behavior. At the origin in the squared momentum variable there are a double pole, a branch-point, and an accumulation of complex first-sheet branch-points. Although the double pole is suggestive of confinement, the existence of acausal complex singularities indicates a deficiency in this first step of the approximation scheme.

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1. INTRODUCTION

This paper is an extension of a previous study[1] of nonperturbative confinement in quarkless quantum chromodynamics, to which we shall refer as I. We continue to explore the hypotheses that (1) it is an indication of confinement for the gluon propagator to be more singular than k^{-3} at small k^2, where k is the gluon four-momentum, and (2) its infrared singularity structure can be properly understood in truncated Dyson–Schwinger (DS) equations. In I we considered a truncated DS equation for the gluon propagator proposed by Mandelstam.[2] Mandelstam worked in the Landau gauge, ignored four-gluon coupling altogether, and moreover he replaced the three-gluon vertex and one of the two internal gluon propagators by bare values. He asserted that the propagator from such a truncated system would behave as k^{-4} at small k^2. We analyzed a somewhat simplified version of Mandelstam's equation and demonstrated (1) that the gluon propagator did have that infrared behavior, and (2) that it also acquired branch-points at complex k^2 in the vicinity of the origin. In fact, such complex branch-points are inconsistent with causality, and causality was used to justify Wick rotation of the internal momentum variable in the truncated DS equation.

It was not clear from I whether the occurrence of unphysical branch-points in the simplified Mandelstam equation was an artifact of additional, somewhat unmotivated assumptions, or whether the full Mandelstam equation [Eq. (2.1) below] would have similar behavior. Here it is shown that solutions of the full Mandelstam equation (however without ghost propagators) have both features of the approximate equation. Namely, the gluon propagator behaves as k^{-4} at asymptotically small k^2, except near the negative real axis, along which complex branch-points seem to accumulate.

Mandelstam justified replacement of the three-gluon vertex function, $\Gamma(p,q,r)$ with $p+q+r=0$, and one gluon propagator, $\Delta(q)$, by their bare values through the Slavnov–Taylor identity for the longitudinal part of the triple-gluon vertex. However, that identity does not require the longitudinal part of Γ to vanish as q and r separately approach zero,[3] so that the cancellation described by Mandelstam is incomplete. Since exact cancellation of propagator and vertex function does not follow from basic principles, the equation obtained by Mandelstam might be expected to be somewhat unphysical.

In contrast to the situation in quantum electrodynamics, the vacuum polarization tensor in quantum chromodynamics is a gauge-dependent entity. Consequently, the behavior of the gluon propagator at small k^2 does not provide direct evidence of confinement. Indeed, a second-order pole in the gluon propagator can be removed by a singular gauge transformation. Our expectation is that the gauge transformation, while removing the pole, will preserve the general feature that propagation of low-frequency modes of the gluon field is suppressed, as is indicative of confinement.

An alternative treatment of DS esquations in QCD has been proposed and examined by Baker *et al.*,[4] and further simplified by Schoenmaker.[5] In this work, an axial gauge is used, so that ghost fields are uncoupled, and may thus be neglected. The basic idea is an *ansatz* for the longitudinal part of the three-gluon vertex, in terms of the full propagator, such that the vertex Slavnov–Taylor identity is satisfied. Within this framework, it is possible to project out the four-gluon terms, so that a closed equation for the propagator results. This has a more complicated nonlinear structure than that of Mandelstam's equation; but there is some reason to hope that the approximation of Baker *et al.* is better than that of Mandelstam.

Baker *et al.* demonstrate that a double pole is a consistent infrared *ansatz*; and they obtain an approximate numerical solution at all energies. However, this work by no means demonstrates that a solution actually exists, much less that it has the required infrared behavior. The point is not merely academic, for Delbourgo has shown that his elegant spectral *ansatz* yields a *nonconfining* infrared behavior,[6] a result that has been confirmed by Khelashvili.[7] Delbourgo also used an axial gauge, and the spectral *ansatz* for the three-gluon vertex is motivated by means of the Slavnov–Taylor (ST) identity. Since it is not expected that a transverse

[a)] Permanent address: Illinois Institute of Technology, Chicago, Illinois 60616.

part of the gluon vertex dominates the infrared, the conflicting claims regarding the behavior of the two approximations, which have the same longitudinal part (in the sense that they are both consistent with the vertex ST identity), is suspect. A careful mathematical treatment of both equations is required, and we hope to provide this in the future.

In Sec. 2 we describe a consistent regularization procedure for Mandelstam's equation [Eq. (2.1) below]. It is reduced to a nonlinear integral equation suitable for analysis [Eq. (2.16)]. The existence of a solution of (2.16), which is analytic in k^2 in a heart-shaped region not including the negative real axis, is established in Sec. 3. A numerical solution for the gluon propagator and procedures for stable analytic continuation are described in Sec. 4. In particular, the existence of unphysical complex branch-points is established, and they are located with precision. The numerical work includes an expansion of the gluon propagator at small spacelike momenta, which is described in Sec. 4 and shown in the Appendix to be an asymptotic expansion.

2. MANDELSTAM'S GLUON EQUATION

In I, we sketched Mandelstam's derivation of an integral equation for the unknown function, $F_1(x)$. Now Eq. (2.9) of I, with the pole term removed, can be rewritten

$$\frac{x}{A + xF_1(x)} = 1 - C + Dx + g^2 \times \int_0^x dy \left\{25\left(1 - \frac{y^2}{x^2}\right) - \frac{7}{2}\left(\frac{x}{y} - \frac{y^3}{x^3}\right)\right\} \frac{F_1(y)}{y}, \tag{2.1}$$

where g is proportional to the SU(3) coupling constant, and where

$$C = 25g^2 \int_0^\infty \frac{dy}{y} F_1(y), \tag{2.2}$$

and

$$D = \frac{7}{2} g^2 \int_0^\infty \frac{dy}{y^2} F_1(y). \tag{2.3}$$

In I, the further approximation was made of dropping the $\frac{7}{2}$ terms above, and it was possible then to prove the existence of a solution, $F_1(x)$, that behaves like x as $x \to 0$ (except along the negative real direction). In this paper we improve the treatment by retaining all the above terms.

The first constraint is that, for consistency, C must be equal to unity; but the integral in (2.2) is ultraviolet divergent, and we may regard $C = 1$ as part of the renormalization prescription, as we did in I. The *ansatz* $F_1(x) \sim x$ as $x \to 0$ is no longer consistent, because of the $\frac{7}{2}$ terms, and must be replaced by $F_1(x) \sim x^\alpha$, $\alpha > 1$. However, the left-hand side of (2.1) still goes linearly to zero, and this imposes the constraint $D = 1/A$. In fact, having removed $1 - C$, we can also scale A and g away by the transformations

$$x \to Agx, \quad y \to Agy, \quad F_1(x) \to g^{-1} F_1(x), \tag{2.4}$$

so that

$$G(x) = -\frac{1}{x^3} \int_0^x dy \left\{25\left(1 - \frac{y^2}{x^2}\right) - \frac{7}{2}\left(\frac{x}{y} - \frac{y^3}{x^3}\right)\right\} \frac{F_1(y)}{y}, \tag{2.5}$$

where

$$G(x) = \frac{F_1(x)}{x + x^2 F_1(x)}, \tag{2.6}$$

is a new unknown function, as in Sec. 3 of I. To this equation must be added the global constraint, corresponding to $D = 1/A$, viz.,

$$\frac{7}{2} \int_0^\infty \frac{dy}{y^2} F_1(y) = 1. \tag{2.7}$$

It is remarkable that this integral will turn out to be ultraviolet and infrared convergent. This is a constraint that was missing in the more approximate equation of I; but we shall find that it can be met.

For infrared convergence in (2.5), we would like

$$F_1(x) \sim \gamma x^{\beta - 1}, \tag{2.8}$$

as $x \to 0$, with $\beta > 2$. Then $G(x)$, on the left-hand side of (2.5), behaves like $x^{\beta - 2}$, while the right-hand side behaves in general like $x^{\beta - 4} + O(x^{\beta - 2})$. This is inconsistent unless the coefficient of $x^{\beta - 4}$ vanishes; fortunately this happens if

$$\beta = (31/6)^{1/2} \approx 2.273\,\ldots, \tag{2.9}$$

a result found by Mandelstam. The value of the coefficient γ in (2.8) can only be obtained numerically, with the help of the global condition (2.7), as we shall see in Sec. 4.

The integral equation (2.5) can be reduced to the nonlinear differential equation

$$6x^2 F_1'' + 18xF_1' - 25F_1 = -\frac{1}{6x}[x^5(x^3G)'']'', \tag{2.10}$$

with

$$F_1(x) = \frac{xG(x)}{1 - x^2 G(x)}. \tag{2.11}$$

The independent solutions of the homogeneous equation (the left-hand side equal to zero), are $x^{-1 \pm \beta}$; so (2.10) can be resolved in terms of them by the method of variation of parameters. The result is

$$F_1(x) = \gamma x^{\beta - 1} - \frac{1}{72\beta x} \times \int_0^x \frac{dy}{y}\left[\left(\frac{x}{y}\right)^\beta - \left(\frac{y}{x}\right)^\beta\right][y^5(y^3 G(y))'']'', \tag{2.12}$$

where the correct boundary condition (2.8) is assured by the first term. The differentiations under the integral in (2.12) can be removed by four partial integrations, and we find

$$x^4 G'' + 9x^3 G' + (36 + \tfrac{63}{4}x^2)G = \Sigma, \tag{2.13}$$

where

$$\Sigma(x) = 36\gamma x^{\beta - 2} - \frac{36x^2 G^2(x)}{1 - x^2 G(x)} - \frac{5}{12} x^2 G(x) - \frac{175}{72\beta x^2} \int_0^x dy \left[\left(\frac{x}{y}\right)^\beta - \left(\frac{y}{x}\right)^\beta\right] y^3 G(y). \tag{2.14}$$

Here F_1 has been eliminated in favor of G, by means of (2.11); this gives rise to the nonlinear term in (2.14). The left-hand side of (2.13) comes from the boundary terms in the partial integrations, except that part of the term proportional to $x^2 G$ has been transferred to the right-hand side [namely the term

$-\frac{1}{12}x^2G(x)$ in (2.14)]. The reason for this transposition is as in I, namely that (2.13) can now be resolved in terms of elementary functions, and the linear term in (2.14) will cause no trouble for small x, thanks to the factor x^2.

The homogeneous equation (2.13) (i.e., with the right-hand side equal to zero) is solved by the functions

$$x^{-7/2}\exp[\pm 6i/x]; \tag{2.15}$$

so (2.13) can be resolved by variation of parameters, the result being

$$G(x) = -\tfrac{1}{6}x^{-7/2}\int_0^x dy\, y^{3/2}\sin\Big(\frac{6}{x}-\frac{6}{y}\Big)\Sigma(y). \tag{2.16}$$

No homogeneous terms may be added. In the next section, we will show that a locally unique solution of (2.16) exists, that is analytic in a certain domain of the x plane, much as in I.

3. EXISTENCE PROOF

To show that Eq. (2.16) has a solution, it is convenient to make these transformations of variables:

$$\xi = \frac{6}{x};\ \tilde G(\xi) = G(x);\ \zeta = \frac{6}{y}-\frac{6}{x}. \tag{3.1}$$

Equation (2.16) takes the form

$$\tilde G(\xi) = P(\tilde G,\xi) \equiv f(\xi) - \xi^{7/2}\int_0^\infty d\zeta \frac{\sin\zeta}{(\xi+\zeta)^{11/2}}\tilde\Omega(\xi+\zeta), \tag{3.2}$$

where

$$f(\xi) = \xi^{7/2}6^{\beta-4}\gamma\int_0^\infty d\zeta\frac{\sin\zeta}{(\xi+\zeta)^{\beta+3/2}}, \tag{3.3}$$

and

$$\tilde\Omega(\zeta) = -\frac{5}{12}\tilde G(\zeta) - \frac{36\tilde G^2(\zeta)}{1-36\zeta^{-2}\tilde G(\zeta)} + \frac{175}{72\beta}\zeta^4\int_\zeta^\infty\frac{d\kappa}{\kappa^5}\Big[\Big(\frac{\zeta}{\kappa}\Big)^\beta - \Big(\frac{\kappa}{\zeta}\Big)^\beta\Big]\tilde G(\kappa). \tag{3.4}$$

We shall establish that (3.2) has a solution $\tilde G(\xi)$ which is analytic in ξ in the domain $\mathscr{D}$, where

$$\mathscr{D}(\rho,\delta) = \left\{\begin{matrix} |\xi|>\rho^{-1}, & \mathrm{Re}\xi\geqslant 0 \\ \dfrac{|\mathrm{Im}\xi|-\rho^{-1}}{|\mathrm{Re}\xi|}>\tan\delta, & \mathrm{Re}\xi<0\end{matrix}\right\}. \tag{3.5}$$

The positive parameters ρ and δ are to be fixed later. The domain $\mathscr{D}$ is the same as that considered in I in connection with proof of existence of a solution of an equation very similar to (3.2). The analysis here is quite parallel to that presented in I.

Let $\mathscr{B}$ be the Banach space of functions analytic in $\mathscr{D}$, with the supremum norm

$$\|f\| = \sup_{\xi\in\mathscr{D}}|f(\xi)|. \tag{3.6}$$

Define the ball $\mathscr{S}$ in the Banach space $\mathscr{B}$ by

$$\mathscr{S} = \{\tilde G\,|\,\tilde G\in\mathscr{B} \text{ and } \|\tilde G\|\leqslant b\}. \tag{3.7}$$

The domain $\mathscr{D}$ has the feature that if ξ lies in $\mathscr{D}$, then so does $\xi+\zeta$, for $\zeta\geqslant 0$. Furthermore, if the constraint

$$36\rho^2 b<1 \tag{3.8}$$

is satisfied, the function $\tilde\Omega(\xi+\zeta)$ is analytic in ζ throughout $\mathscr{D}$ when $\zeta\geqslant 0$, and the integral in (3.2) converges uniformly to a function analytic in $\mathscr{D}$. The inhomogeneous term in (3.2), $f(\xi)$, can be shown by an analysis similar to that of Appendix B of I to be analytic in the ξ plane cut along the negative real axis. Consequently, $P(\tilde G,\xi)$ is analytic for ξ in $\mathscr{D}$ if (3.8) is met.

We shall show that P maps the ball into itself and is a contraction mapping, if suitable constraints are placed upon ρ, δ, and b. The Banach contraction mapping theorem may then be applied to give a solution of the equation

$$\tilde G(\xi) = P(\tilde G,\xi), \tag{3.9}$$

which is unique in the ball $\mathscr{S}$ of $\mathscr{B}$.

By using condition (3.8), one obtains the following bound upon $\Omega(\zeta)$ for $\zeta\in\mathscr{D}$:

$$|\Omega(\zeta)|\leqslant\Big(\frac{5}{12}+\frac{175}{36(16-\beta^2)}\Big)b + \frac{36b^2}{1-36\rho^2 b}\equiv J(\rho,b). \tag{3.10}$$

One may obtain the following bound directly from (3.3):

$$|f(\xi)| \leqslant \frac{\gamma}{36}(6\rho)^{\beta-2}\Big\{1+\Big(\beta+\frac{3}{2}\Big)\int_0^\infty\frac{d\omega}{|\omega-e^{i\epsilon}|^{\beta+5/2}}\Big\}\equiv C_\epsilon, \tag{3.11}$$

where $|\arg\xi|\leqslant\pi-\epsilon$. Using (3.10) and (3.11) in (3.2), we obtain

$$|P(\tilde G,\xi)|\leqslant C_\epsilon + \rho J(b,\rho)D_\epsilon, \tag{3.12}$$

where

$$D_\epsilon = \int_0^\infty\frac{d\omega}{|\omega-e^{i\epsilon}|^{11/2}}. \tag{3.13}$$

Consequently, the ball $\mathscr{S}$ is mapped into itself by P if

$$C_\epsilon<b, \tag{3.14}$$

and

$$\rho\leqslant\frac{b-C_\epsilon}{J(b,\rho)D_\epsilon}. \tag{3.15}$$

The contractivity condition is

$$\|P(G_1)-P(G_2)\|\leqslant K\|G_1-G_2\|, \tag{3.16}$$

with K less than 1, for any functions G_1 and G_2 in the ball $\mathscr{S}$.

To obtain an estimate on the difference of the nonlinear terms in P, it is convenient to introduce

$$\Sigma(\tilde G,\zeta) = \frac{36\tilde G^2(\zeta)}{1-(36/\zeta^2)\tilde G(\zeta)}. \tag{3.17}$$

The derivative of this algebraic function with respect to $\tilde G$ is well-defined, and for $\tilde G$ in $\mathscr{S}$ and ζ in $\mathscr{D}$ it is subject to the bound

$$\Big|\frac{d\Sigma}{d\tilde G}\Big|\leqslant\frac{108b}{(1-36b\rho^2)^2}\equiv L(b,\rho). \tag{3.18}$$

One may then use this constraint, along with the mean value theorem, to obtain

$$|\Sigma(G_1,\zeta)-\Sigma(G_2,\zeta)|\leqslant L(b,\rho)|G_1(\zeta)-G_2(\zeta)|. \tag{3.19}$$

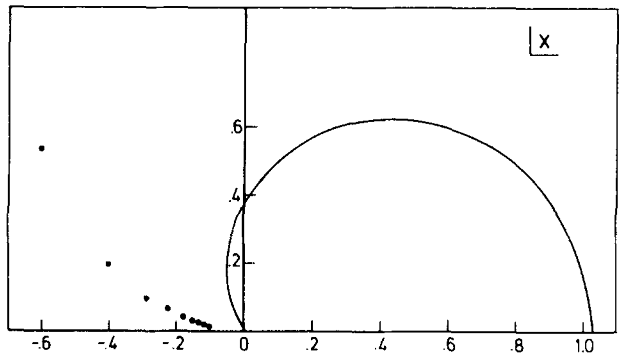

FIG. 1. The cardioid region to which the Banach theorem applies. The points outside this region are the locations of branch-points determined as described in Sec. 4.

Using (3.19) and making more direct estimates of the other terms in P, one obtains an estimate of the form (3.16), with

$$K=\rho\left[L(b,\rho)+\frac{5}{12}+\frac{175}{36(16-\beta^2)}\right]D_\epsilon. \tag{3.20}$$

Consequently, the mapping is contractive if

$$K<1. \tag{3.21}$$

The conditions for a contraction mapping, (3.8), (3.15), and (3.21), may simultaneously be met for any number ϵ between 0 and $\pi/2$. Because the integrals C_ϵ and D_ϵ depend upon ϵ, the maximal values of the parameters ρ, δ, and b also depend upon it. The function $\tilde{G}(\xi)$, obtained as the locally unique fixed point of Eq. (3.2) in each of the domains $\mathscr{D}(\rho,\epsilon)$, is analytic in ξ in the union of these domains. We have extended this fixed-point proof to a set of domains in the right-half x plane, which are sectors of varying radius and opening angle that are symmetric about the real axis. The full domain of analyticity in the variable x, which is obtained numerically as the union of the regions in which conditions (3.8), (3.15), and (3.21) are met, is shown in Fig. 1. The parameter γ is chosen so that condition (2.7) is met [see Eq. (4.10) below].

4. NUMERICAL ANALYSIS

We have shown that the integral equation (2.16) has a solution $G(x)$ which is bounded and analytic in the heart-shaped domain $\mathscr{D}$, with the asymptote $\gamma x^{\beta-2}$ as x approaches zero within $\mathscr{D}$. We wish to obtain this solution numerically, and thereby determine the behavior of $G(x)$ outside the domain $\mathscr{D}$. Equation (2.16) is a well-behaved functional equation for G—at least so long as x is in $\mathscr{D}$—but it seems impractical to attempt a direct global solution of (2.16). Instead, we have chosen to determine $G(x)$ in some domain of small x from (2.16), and then to get G elsewhere by solving a differential equation such as (2.10), which is equivalent to (2.16).

We obtain $G(x)$ at small x within $\mathscr{D}$ by developing an asymptotic series for x in that region. Although we justify the asymptotic series by analysis of Eq. (2.16), the series itself is most easily developed from the integro-differential system (2.13) and (2.14). One may make a consistent expansion in powers of both x^2 and x^β as follows:

$$G(x)=\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}a_{jn}x^{j\beta+2n-4}. \tag{4.1}$$

The coefficient of the leading term is $a_{11}=\gamma$, and the higher-order terms may be determined recursively from these formulas:

$$a_{j,1}=-\gamma a_{j-1,1}\quad \text{for } j>1; \tag{4.2}$$

$$a_{1,n}=-c_{1,n-1}\quad \text{for } n>1; \tag{4.3}$$

$$a_{j,n}=-\gamma a_{j-1,n}-c_{j,n-1}+\sum_{j'=1}^{j-1}\sum_{n'=1}^{n-1}a_{j'n'}c_{j-j',n-n'}\quad \text{for } j,n>1. \tag{4.4}$$

We have used the array c_{jn}:

$$c_{jn}=\frac{a_{jn}}{36}\left[(2n+j\beta)^2+\frac{1}{6}+\frac{175}{36}\frac{1}{(2n+j\beta)^2-\beta^2}\right]. \tag{4.5}$$

We show in the Appendix that (4.1) is indeed an asymptotic series for $G(x)$ by truncating it to include only powers of x not greater than M. Our estimate for the difference between $G(x)$ and the truncated series depends upon M, as well as the location of the point x (its amplitude *and* phase) in the domain $\mathscr{D}$. In practice, for a given x_0 we truncate the series (4.1) so that the computed values of $G(x_0)$ and its first four derivatives give least discrepancy in the fourth-order differential equation (2.10). We can achieve single-precision accuracy for $G(x)$ (order 10^{-12}) on the CDC Cyber 160/170 computer in Groningen at small x in $\mathscr{D}$ with M of order 20; for $\gamma=0.0608$ we can use the series on the real x axis out to about 0.13, and less far in complex directions. The values of G and its first three derivatives are used as a starting point for solution of (2.10).

Let us consider the solution of the fourth-order nonlinear differential equation (2.10) from starting values of G and its first three derivatives at a point $x_0\neq 0$. If the values are such that $x_0^2G(x_0)\neq 1$, the fourth derivative can be determined from (2.10). Furthermore, from the general theory of differential equations involving analytic functions of both the dependent and independent variables,[8,9] one expects there to be a locally unique solution $G(x)$ corresponding to these initial data, which is analytic in x in some neighborhood of x_0. Of course, the solutions that develop from different initial data bear no simple relation to one another, because of the nonlinearity in G. The singularities of a solution of (2.10) may be of two types: (1) "fixed singularities" at $x=0$ and $x=\infty$, and (2) "movable singularities" at points for which

$$x^2G(x)=1. \tag{4.6}$$

The point $x=0$ is an irregular singular point of the differential equation, and one expects $G(x)$ to have an essential singularity at that point, with possibly nontrivial Riemann sheet structure as well. The locations of the movable singularities depend upon the initial data. There is no simple prescription to determine the locations of these movable singularities from the initial data; in general one must resort to numerical analysis.

It is consistent with the integro-differential system (2.13) and (2.14), and therefore with (2.10), for $G(x)$ to have the following asymptotic form near a branch point at $x = d$, at which (4.6) is satisfied:

$$G(x) \sim \frac{1}{d^2} \pm \frac{6}{2^{1/2} d^4}(x-d)\left[\ln\frac{(x-d)}{d_0}\right]^{1/2}, \qquad (4.7)$$

where d_0 is a constant. With this asymptotic form, for which $G'(x)$ diverges logarithmically as x approaches d, the most singular terms in the system cancel near $x = d$. This divergence of G' and the higher derivatives in the vicinity of the branch points makes it difficult to locate them numerically by direct solution of (4.6).

The solution of (2.16) described in Sec. 3 is one of an infinite number of solutions of the differential equation (2.10). Furthermore, we expect from the general theory of analytic differential equations that it is the only solution of (2.10) with the asymptote $\gamma x^{\beta-2}$ at small positive x, so that all other solutions are so singular as to be inconsistent with the original integral equation (2.5) in that region. In the fixed-point proof for existence of a solution $G(x)$, analytic in $\mathscr{D}$, it was important to ensure that condition (4.6) was not met anywhere in $\mathscr{D}$, so that the movable singularities are avoided in that domain.

We shall construct the function $G(x)$ and effect its analytic continuation outside $\mathscr{D}$ by numerical means. One would hope for physical reasons that $G(x)$, being related to the full gluon propagator in Mandelstam's truncation of Dyson–Schwinger equations in quantum chromodynamics, would turn out to be analytic on the physical sheet of the cut x plane, with a branch-cut lying only along the negative real x axis, and bounded at infinity in that plane. However, we have no analytical control over the behavior of G outside $\mathscr{D}$, and must resort to numerical procedures to determine its analytic structure. The real constant γ must be chosen so that the integral condition (2.7) is met by $F_1(x,\gamma,G)$. Strictly speaking, since G is not guaranteed by our analysis to have a continuation to the full positive real axis, the integral (2.7) need not even exist. Our procedure for choosing γ requires numerical work for its justification.

With initial data obtained from the asymptotic series (4.1), the differential equation (2.10) is integrated from a starting point x_0 by an explicit fourth-order Runge–Kutta routine, in which it is considered as four coupled first-order differential equations for G, G', G'', and G'''. For a discussion of this standard procedure, see Refs. 10 and 11. The step length Δx is changed with changing x to maintain accuracy. In particular, it is necessary to take rather small steps when x is small, or when $x^2G(x)$ is close to $+1$. When one is near $x = 0$, or near a movable singularity, or both, instabilities are apt to creep in. There may be no immediate suggestion of inaccuracy, since cumulative errors are equivalent to changes in the values of G and its first three derivatives at the starting point. We have tested the integration routine to be certain that the values of $G(x)$ are indeed path-independent and stable away from the fixed and movable singularities.

The integral (2.7) is computed over small x, $0<x<0.1$, by using the asymptotic series (4.1). For $x>0.1$ we determine the integral

$$I(x) = \frac{7}{2}\int_0^x \frac{dy}{y}\,\frac{G(y)}{1-y^2G(y)}, \qquad (4.8)$$

by solving the equivalent differential equation

$$\frac{dI}{dx} = \frac{7}{2x}\,\frac{G(x)}{1-x^2G(x)}. \qquad (4.9)$$

The differential equation (4.9) for $I(x)$ is incorporated in the Runge–Kutta integration procedure to determine $G(x)$. The constraint (2.7), $I(\infty) = 1$, is satisfied by choosing the parameter γ to be

$$\gamma = 0.060\,870\,966\,1 \pm 0.000\,000\,000\,1. \qquad (4.10)$$

As an independent check of the accuracy of this result, we have verified that the ratio of the change in $I(\infty)$ to the change in γ,

$$\Delta I(\infty)/\Delta\gamma \approx 20.17, \qquad (4.11)$$

is numerically stable down to $\Delta\gamma$ of 10^{-10}. It is important for the asymptotic series to give an accurate representations of $I(x)$ at small x, since more than 40% of the integral comes from x below 0.1.

With the choice (4.10) for γ, the function $x^2G(x)$ is analytic in the right half x plane, approaches $+1$ at infinity in the right half-plane, and is monotonically increasing in x for real positive x. The behavior of the corresponding function $F_1(x)$ is shown in Fig. 2. This function has the following asymptote at large real x:

$$F_1(x) = \frac{1}{[50\ln(x/x_0)]^{1/2}}, \qquad (4.12)$$

as required for consistency with (2.5).

For exploring the behavior of $G(x)$ in the left half x plane, expecially at small x, it is quite useful to be able to integrate the differential equation (2.10) along implicitly defined contours that are determined as we go along. For example, to keep the magnitude of $G(x)$ constant to first order in step size Δx, one must require

$$\Delta\,[G^*(x)G(x)] = O(\Delta x)^2 \qquad (4.13)$$

or

$$\mathrm{Re}[G^*(x)G'(x)\Delta x] = 0. \qquad (4.14)$$

At each step of the Runge–Kutta routine we choose the *phase* of Δx so that (4.14) is met, using values of G and G' at the current position. Actually, it is advantageous to keep

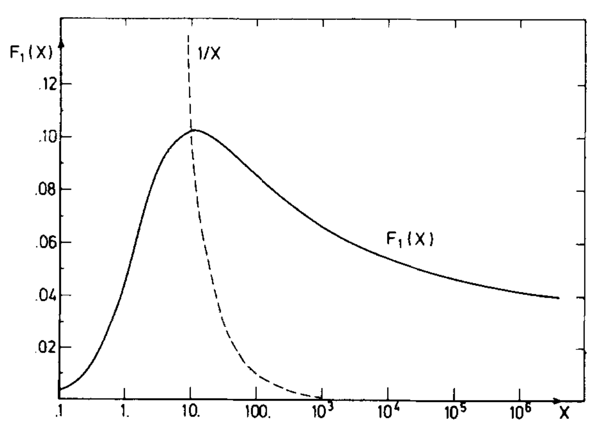

FIG. 2. Graph of $F_1(x)$ versus x. The other term in (4.21), $1/x$, is shown for reference.

$|G(x)|$ constant only to first order in step size, since the slight, gradual changes in G provide a good monitor on the level of accuracy with which the function $G(x)$ is being determined. It is even more useful to integrate along a contour on which $|x^2G(x)|$ is held roughly constant, to keep a safe distance away from the movable singularities of (2.10). The corresponding condition on the phase of Δx is

$$\mathrm{Re}\left\{G^*(x)\left[G'(x)+\frac{2}{x}G(x)\right]\Delta x\right\}=0. \qquad (4.15)$$

With thorough analysis and testing, we have made a stable extrapolation of $G(x)$ into the left half x plane. We find that, when γ is given by (4.10), there are branch-points at locations given in Table 1.

It is consistent to suppose that there is an infinite number of branch-points on the physical sheet, accumulating at $x=0$ near the negative real axis, but such a hypothesis cannot be tested numerically. Of course, it is reasonable to expect that $x^2G(x)$ takes on the value $+1$ at an infinite number of points near the essential singularity at $x=0$, but we have found no general argument to indicate that such points must lie on the physical Riemann sheet. We have no information on the asymptotic behavior of $G(x)$ as x approaches zero, except when x is in $\mathscr{D}$.

Since it is essentially a numerical problem to prove the existence of branch-points of $G(x)$ and to locate them, it is appropriate to give the following information concerning the accuracy with which G is determined:

1). At $x_0=(-0.5,0.75)$, the function $G(x)$ is reliably determined to be (0.324 361 862 88, 0.142 874 479 38), with the error in the last digit.

2). When Eq. (2.10) is started from x_0 and integrated counterclockwise around a square contour with sides -0.25, and $0.25\,i$, respectively, the total change in the real and imaginary parts of G is less than 10^{-11}.

3). By contrast, when Eq. (2.10) is started from x_0 and integrated counterclockwise around a square contour of sides -0.25 and $-0.25\,i$, respectively, the new value of G is ($+6.145\,867\,784\,1$, $-0.386\,126\,067\,6$), with the error in the last digit.

4). The results in 2). and 3). are valid for 1000, 2000, and 4000 steps per side in the Runge–Kutta integration.

This information is our basis for concluding that a branch-point lies inside the second square, but not in the first; see Table I.

TABLE I. Location of first nine branch-points of $G(x)$.

n	Rex	Imx
1	− 0.601 22	± 0.535 25
2	− 0.403 17	± 0.191 20
3	− 0.289 55	± 0.098 45
4	− 0.224 28	± 0.060 43
5	− 0.182 57	± 0.041 08
6	− 0.153 76	± 0.029 86
7	− 0.132 71	± 0.022 74
8	− 0.116 71	± 0.017 94
9	− 0.104 11	± 0.014 54

It is a nontrivial numerical problem to maintain accuracy while getting close enough to branch-points to be able to find and isolate them, especially at small x, where the branch-points themselves are close together and other singularities are nearby. We have found it rather efficient to integrate (2.10) along a curve for which $|x^2G(x)|$ is fixed at a value somewhat less than 1. The phase of $x^2G(x)$ changes continuously along such a curve, and one is fairly close to a branch-point whenever $x^2G(x)$ becomes real and positive. The branch-points are located more precisely by integrating along closed paths enclosing successively smaller regions. The branch-points can be determined quite accurately by using steps determined by solving (4.6) through Newton iteration. Even though $G'(x)$ diverges logarithmically at the branch-point, according to (4.7), the method works rather well.

A direct numerical solution of (2.10) is subject to criticism on the grounds that is has solutions which are very singular at small x, but reasonably well-behaved elsewhere, and cumulative errors will, in effect, switch us over to one of the unacceptable solutions as we change x. We avoid this problem to a great extent by starting at small x in $\mathscr{D}$ using the asymptotic series (4.1), thereby assuring that at the outset there is very little contamination of the solution. Correspondingly, we expect a substantial loss in precision when we attempt to integrate from large to small $|x|$.

An alternate procedure is to solve the integro-differential equations (2.13) and (2.14). We can write them as a coupled system of equations for $G(x)$, $\Omega_1(x)$, and $\Omega_2(x)$; Ω_1 and Ω_2 being defined as

$$\Omega_1(x)=\frac{1}{x^2}\int_0^x dy\,y^3\left(\frac{y}{x}\right)^\beta G(y), \qquad (4.16)$$

$$\Omega_2(x)=\frac{1}{x^2}\int_0^x dy\,y^3\left(\frac{x}{y}\right)^\beta G(y). \qquad (4.17)$$

The coupled system is

$$\begin{aligned}G''(x)= &-\frac{9}{x}G'(x)-\frac{97}{6x^2}G(x)\\ &+\frac{1}{x^4}\left[\frac{175}{72\beta}(\Omega_1(x)-\Omega_2(x))\right.\\ &\left.+36\left(\gamma x^{\beta-2}-\frac{G(x)}{1-x^2G(x)}\right)\right], \end{aligned}\qquad (4.18)$$

$$\Omega_1'(x)=xG(x)-\frac{2+\beta}{x}\Omega_1(x), \qquad (4.19)$$

$$\Omega_2'(x)=xG(x)-\frac{2-\beta}{x}\Omega_2(x). \qquad (4.20)$$

The leading asymptotic term for $G(x)$,$\gamma x^{\beta-2}$, appears explicitly in this system of equations.

We have used a fourth-order Runge–Kutta routine to solve the system (4.18)–(4.20), which we treat as coupled first-order equations for G, G', Ω_1, and Ω_2. The results are virtually identical with those obtained by solving (2.10) for x not near zero, and the coupled system has virtually the same degree of instability at small x as (2.10). Although there is one solution of this system of equations which is well-behaved in $\mathscr{D}$, there is an infinite class of other solutions that are not, and cumulative uncertainties will surely lead to nu-

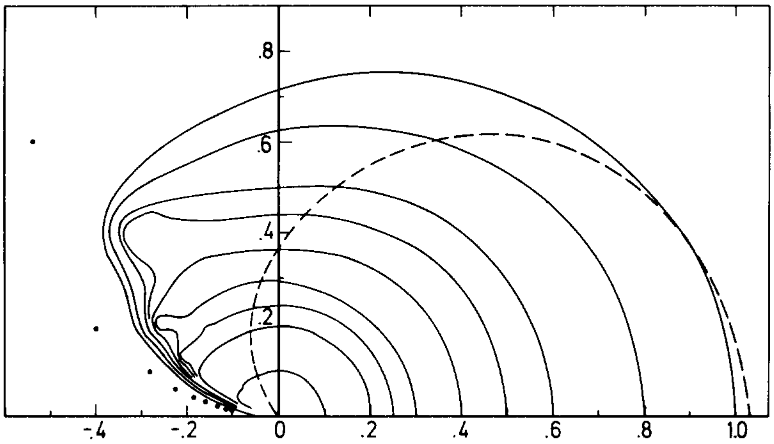

FIG. 3. Contours of constant magnitude of $x^2G(x)$ are shown. The values of $|x^2G|$ on successively larger contours are: 0.0003, 0.0015, 0.0025, 0.0034, 0.0070, 0.0112, 0.0161, 0.0282, 0.0423. The points give the locations of branch-points, at which $x^2G(x) = 1$.

merical instabilities here, just as they did with (2.10). In fact, one might expect that any replacement of (2.16) by a system of differential equations would behave in a similar fashion.

In Fig. 3 we have shown the contours in the upper half x plane along which $x^2G(x)$ is of constant magnitude, with γ given by (4.10). These contours are determined numerically from points that begin on the positive real axis. The contours become closer together in the vicinity of the branch-points in the second quadrant, and they all seem to approach the origin from the negative real direction. The large region between contours near $(-0.3,0.4)$ occurs because the derivative of $x^2G(x)$ has a zero in that region. The contours in Fig. 3 are numerically stable.

In Fig. 2 the function $F_1(x)$, which is given in terms of $G(x)$ by (2.11), is plotted for real x. The function has the asymptote (2.8) at small x, and the asymptote (4.12) at large x. The function

$$F(q^2) = q^{-2} + F_1(q^2) \tag{4.21}$$

is the factor multiplying the free-gluon propagator to give the full propagator in Mandelstam's equation. The physical scale for the momentum q^2 cannot be determined from the DS equation itself, but must be fixed by additional information, such as locations of gluonium states.

The solution of the full Mandelstam equation (2.1) is seen to have behavior similar to that obtained in I for the approximate case, and to suffer from the same deficiency, namely the appearance of branch-points at complex q^2. They must be regarded as an intrinsic deficiency of the Mandelstam equation, which one would hope to be able to eliminate by making a less drastic truncation of Dyson–Schwinger equations.

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APPENDIX

Let us truncate the series (4.1) in such a way that only those terms are included for which the powers of x are less than, say, M; we call this truncated expression $G_M(x)$. We shall show that

$$\lim_{|x|\to 0} |x|^{-M}|G(x) - G_M(x)| = 0 \tag{A1}$$

for $x \in \mathscr{D}(\rho,\epsilon)$. This is the natural generalization of the concept of an asymptotic series[12] to the case in which nonintegral powers occur. Set

$$R_M(x) = x^4G_M'' + 9x^3G_M' + (36 + \tfrac{63}{4})G_M - \Sigma(x,G_M), \tag{A2}$$

where Σ was defined in (2.14). For a given M, $G_M(x)$ is bounded for $x \in \mathscr{D}(\rho,\delta)$, and we can certainly find a subdomain, $\mathscr{F}_M \subset \mathscr{D}(\rho,\delta)$, for which say,

$$|x^2G_M(x)| \leqslant \tfrac{1}{2}. \tag{A3}$$

Now $[1 - x^2G_M(x)]R_M$ can be written as a finite number of terms, involving powers between x^M and x^{2M+2} and hence, in view of (A3),

$$|R_M(x)| \leqslant K_M|x|^M \tag{A4}$$

for $x \in \mathscr{F}_M$, where K_M depends on M. One may integrate (A2) to obtain an equation for G_M which is similar to (2.16), with an extra inhomogeneous term from R_M:

$$G_M(x) = -\tfrac{1}{6}x^{-7/2}\int_0^x dy\, y^{3/2} \times \sin\left(\frac{6}{x} - \frac{6}{y}\right)(\Sigma(y,G_M) + R_M(y))\,. \tag{A5}$$

Let us subtract (A5) from (2.16), and express the result in terms of the function

$$h(x) = G(x) - G_M(x) \tag{A6}$$

as

$$h(x) = -\tfrac{1}{6}x^{-7/2}\int_0^x dy\, y^{3/2}\sin\left(\frac{6}{x} - \frac{6}{y}\right) \times [\Sigma(y,G) - \Sigma(y,G-h) + R_M(y)]. \tag{A7}$$

Equation (A7) is treated as a nonlinear integral equation for h, with the function G taken as the solution of (2.16) described in Sec. 3. The term $36\gamma x^{\beta-2}$ cancels out of (A7), so that R_M provides the only inhomogeneity. By an analysis similar to that described in Sec. 3, it is a simple exercise to establish the existence of a solution, $h(x)$, which is analytic in the domain $\mathscr{F}_M$, and in that region subject to the bound

$$|h(x)| < K'_M |x|^M, \tag{A8}$$

with the constant K'_M dependent upon M. The result, which may also be written as

$$|G(x) - G_M(x)| < K'_M |x|^M, \tag{A9}$$

guarantees that (4.1) is indeed an asymptotic series for G. For the simplified case considered in I, the corresponding series was not strongly asymptotic, and one would not expect that property of (4.1).

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